Soit $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3 \,/\, \vec{u} \wedge \vec{v} = \vec{w}$. Montrons que $\vec{w} = \begin{pmatrix} u_y v_z - v_y u_z \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{pmatrix}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}.$$

<u>Lemme 1</u> (Définition)

$$\vec{u} \wedge \vec{v} = \begin{cases} \vec{0} \quad \text{si } \exists \lambda \in \mathbb{R},\ \vec{u} = \lambda \vec{v} \\ \text{sinon,} \\ \text{Le vecteur unique } \vec{w} \text{ tel que : } \bullet \langle \vec{w}, \vec{u} \rangle = 0 \text{ et } \langle \vec{w}, \vec{v} \rangle = 0 \\ \qquad \bullet \|\vec{w}\| = \|\vec{u}\| \|\vec{v}\| \sin(\vec{u}, \vec{v}) \\ \qquad \text{(Par définition)} \\ \qquad \bullet \{\vec{u}, \vec{v}, \vec{w}\} \text{ base directe} \end{cases}$$

<u>Lemme 2</u> : Soit $B_0 = \{\vec{i}, \vec{j}, \vec{k}\}$ base de $\mathbb{R}^3$ directe

et $\{\vec{i}, \vec{j}, \vec{k}\} \xrightarrow{\varphi} \{\varphi(\vec{i}), \varphi(\vec{j}), \varphi(\vec{k})\} = \{\vec{u}, \vec{v}, \vec{n}\}$

Alors $\det_{B_0}(\vec{u}, \vec{v}, \vec{n}) = (\vec{u} \wedge \vec{v}) \cdot \vec{n}$

<u>Preuve</u> : soit $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ $\quad \vec{u}, \vec{v} \in \mathbb{R}^3$

$\vec{x} \longmapsto \det_{B_0}(\vec{u}, \vec{v}, \vec{x})$

$= f(\vec{x})$

D'après le théorème de représentation de Riesz, (f forme linéaire)

$\exists! \, \vec{w} \in \mathbb{R}^3, \forall \vec{x}, f(\vec{x}) = \vec{w} \cdot \vec{x} \Leftrightarrow \det_{B_0}(\vec{u}, \vec{v}, \vec{x}) = \vec{w} \cdot \vec{x}$

<u>Montrons que cet unique $\vec{w} = \vec{u} \wedge \vec{v}$</u>

- si $\vec{u} = \lambda \vec{v}$, $\det_{B_0}(\vec{u}, \vec{v}, \vec{x}) = 0 = \vec{w} \cdot \vec{x}$, $\forall \vec{x} \in \mathbb{R}^3$

$\Rightarrow \vec{w} = \vec{0}$, $\vec{w}$ unique

- <u>$\langle \vec{w}, \vec{u} \rangle = \langle \vec{w}, \vec{v} \rangle = 0$</u> : • Pour $\vec{x} = \vec{u}$, $\vec{w} \cdot \vec{u} = \det(\vec{u}, \vec{v}, \vec{u})$
$= 0$

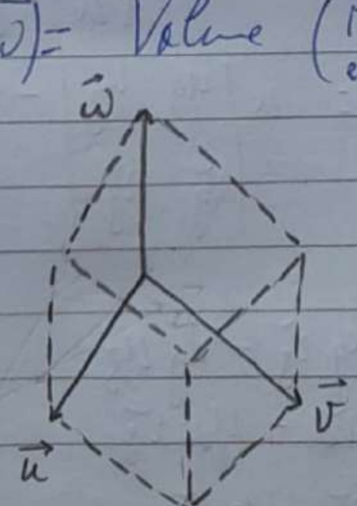
• Pour $\vec{x} = \vec{v}$, $\vec{w} \cdot \vec{v} = 0$
($\vec{w}$ unique)

- si $\vec{x} = \vec{w}$,

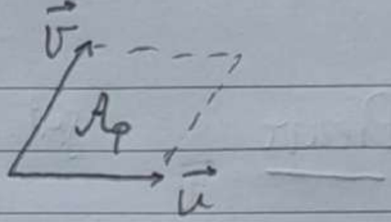
$\det(\vec{u}, \vec{v}, \vec{w}) = \|\vec{w}\|^2$

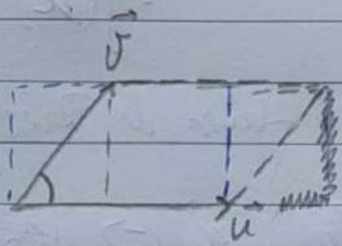
<u>Or</u> $\det(\vec{u}, \vec{v}, \vec{w}) = $ Volume (parallélépipède engendré par $\vec{u}, \vec{v}, \vec{w}$)

$= V_P$



Or $V_p = \|\vec{\omega}\| \cdot \mathcal{A}_{\text{parallélogramme engendré par } \vec{u}, \vec{v}}$





$\mathcal{A}_p =$ Aire rectangle $\begin{pmatrix} l = \|\vec{u}\| \\ L = \|\vec{v}\| \end{pmatrix}$

$= \|\vec{u}\| \, \|\vec{v}\| \sin(\vec{u}, \vec{v})$

$\Rightarrow V_p = \|\vec{\omega}\| \, \|\vec{u}\| \, \|\vec{v}\| \sin(\vec{u}, \vec{v})$

$\Rightarrow \det(\vec{u}, \vec{v}, \vec{\omega}) = \|\vec{\omega}\|^2 = \|\vec{\omega}\| \, \|\vec{u}\| \, \|\vec{v}\| \sin(\vec{u}, \vec{v})$

$\Rightarrow \boxed{\|\vec{\omega}\| = \|\vec{u}\| \, \|\vec{v}\| \sin(\vec{u}, \vec{v})}$

- $\det(\vec{u}, \vec{v}, \vec{\omega}) = \|\vec{\omega}\|^2 > 0$. Alors $[\vec{u}, \vec{v}, \vec{\omega}]$ directe

D'où, $\vec{u} \wedge \vec{v} = \vec{\omega}$ et $\underbrace{\det(\vec{u}, \vec{v}, \vec{n}) = (\vec{u} \wedge \vec{v}) \cdot \vec{n}}_{(*)}$

Pour $\vec{n} = (a, b, c) \in \mathbb{R}^3$, $(*) \Rightarrow \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ a & b & c \end{vmatrix} = \omega_x a + \omega_y b + \omega_z c$

$\Rightarrow \omega_x a + \omega_y b + \omega_z c = a \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix} - b \begin{vmatrix} u_x & u_z \\ v_x & v_z \end{vmatrix} + c \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

$\Rightarrow \begin{cases} \omega_x = u_y v_z - v_y u_z \\ \omega_y = u_z v_x - u_x v_z \\ \omega_z = u_x v_y - u_y v_x \end{cases}$